3.9: Exponential & Logarithmic Functions

**Objective:** Given an exponential or logarithmic function, find its derivative function algebraically.

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**Relationship Between $e^x$ and $\ln x$**

If $y = e^x$, then $x = \ln y$

$e$ is an irrational number equal to 2.71828182845... and is used as a base for natural exponential functions, such as $f(x) = e^x$.

$\ln$ is a natural logarithm with $e$ as its base ($\ln = \log_e$) and is used to determine the exponents of natural exponential functions. Natural logarithmic functions take the form, $f(x) = \ln x$.

Natural exponential functions and natural logarithmic functions—both with respect to $x$—are inverses (i.e. If $f(x) = \ln x$, then $f^{-1}(x) = e^x$; and if $f(x) = e^x$, then $f^{-1}(x) = \ln x$). Therefore, their operations cancel each other out. For example,

\[
\ln e^x = x \quad \text{and} \quad e^{\ln x} = x
\]

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**Natural Exponential Function & Derivative**

\[
\frac{d}{dx}(e^x) = e^x
\]

*Remember to use the Chain Rule when appropriate*

**Example 1: Differentiate a Natural Exponential Function**

Find $f'(x)$ if $f(x) = e^{\cos x}$. Confirm the answer by graphing the algebraic and numerical derivatives on the same screen.
Natural Logarithm & Derivative

\[
\frac{d}{dx} (\ln x) = \frac{1}{x}
\]

The derivative of the natural logarithm function is the reciprocal function.

**Example 2: Differentiate a Natural Logarithm Function**

Find \( f'(x) \) if \( f(x) = \ln x^3 \). Confirm the answer by comparing values with the algebraic and numerical derivatives.

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<thead>
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<th>( x )</th>
<th>( f(x) = \ln x^3 )</th>
<th>( f'(x) )</th>
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<tr>
<td>10</td>
<td>0.3</td>
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</table>

Other Exponential Functions & Derivative

\[
\frac{d}{dx} (b^x) = b^x \ln b
\]

**Example 3: Differentiate an Exponential Function**

Find \( f'(x) \) if \( f(x) = 10(1.05^x) \).

Other Logarithmic Functions & Derivative

Recall that a logarithm is an exponent. (e.g., if \( 3^x = 9 \), then the exponent \( x = \log_3 9 \).)

**Definition of Logarithm:**

\( a = \log_b c \) if and only if \( b^a = c \), \( b > 0 \), and \( b \neq 1 \), where \( b \) is the base, \( a \) is the exponent, and \( c \) is the “answer” to \( b^a \).
Properties of Logarithmic Functions:

- **Log of a Power**: \( \log_b (c^d) = d \cdot \log_b c \)
- **Log of a Product**: \( \log_b (cd) = \log_b c + \log_b d \)
- **Log of a Quotient**: \( \log_b \left( \frac{c}{d} \right) = \log_b c - \log_b d \)

**Derivative of a Logarithmic Function:**

\[
\frac{d}{dx} (\log_b x) = \frac{1}{x \ln b}
\]

**Example 4: Differentiate a Logarithmic Function**

Find \( f'(x) \) if \( f(x) = \log_4 (x^2 - 9x) \)

**Example 5: Using the Logarithmic Properties to Differentiate**

Find \( f'(x) \) if \( f(x) = \log_5 \left( \frac{2x+1}{x^2+3} \right) \)

**Example 6: CHALLENGE**

Find \( f'(x) \) if \( f(x) = [\log_4 (1 + e^x)]^2 \)